# Схема Кранка-Николсона. семинар 5



## Пример 1

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} + t + e^x \qquad u(t=0, x) = e^t$$

1) записать схему Кранка-Николсона
2) привести к виду, удобному для прогонки

$$\begin{cases} u(t, x=0) = t+1 \\ \dfrac{\partial u}{\partial x}(t, x=1) = 2{,}7 + t \end{cases}$$

3) проверить сходимость прогонки
4) вывести $\alpha_j, \beta_j$
5) получить $\alpha_1, \beta_1$ из ЛГУ.
6) Найти $u^{n+1}_{N_x}$ из П.Г.У
7) записать рекуррентное прогоночное соотношение
8) записать блок-схему

1) $$\frac{u_j^{n+1} - u_j^n}{\Delta t} = 2\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} + 2\frac{u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n}}{h^2} + n\Delta t + e^{(j-1)h}$$

2) $$u_{j+1}^{n+1}\underbrace{\left[\frac{-2\Delta t}{h^2}\right]}_{a_j} + u_j^{n+1}\underbrace{\left[1 + \frac{4\Delta t}{h^2}\right]}_{b_j} + u_{j-1}^{n+1}\underbrace{\left[\frac{-2\Delta t}{h^2}\right]}_{c_j} =$$

$$= \underbrace{u_j^n + \frac{2\Delta t}{h^2}\cdot\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + \Delta t\cdot\left[n\Delta t + e^{(j-1)h}\right]}_{\xi_j^n}$$

3) условие сходимости:

$$|a_j| + |c_j| < |b_j|$$

$$\frac{4\Delta t}{h^2} < 1 + \frac{4\Delta t}{h^2}$$ ✓

4) $U_{j+1}^{n+1} \cdot [a_j] + U_j^{n+1} [b_j] + U_{j-1}^{n+1} [c_j] = \delta_j^n$

$$U_{j-1}^{n+1} = \alpha_{j-1}^{n+1} \cdot U_j^{n+1} + \beta_{j-1}$$

видим связь, подставляем:

$$\underline{\underline{U_{j+1}^{n+1}}} [a_j] + \underline{U_j^{n+1}} [b_j] + [c_j] \cdot [\alpha_{j-1} \cdot \underline{U_j^{n+1}} + \beta_{j-1}] = \delta_j^n$$

приводим виду прогоночного соотношения

$$\underline{U_j^{n+1}} = \alpha_j \underline{\underline{U_{j+1}^{n+1}}} + \beta_j$$

Тогда:

$$U_j^{n+1} = -\frac{a_j}{b_j + c_j \cdot \alpha_{j-1}} \cdot U_{j+1}^{n+1} + \frac{\delta_j^n - c_j \cdot \beta_{j-1}}{b_j + c_j \alpha_{j-1}} \Rightarrow$$

$$\alpha_j = -\frac{a_j}{b_j + c_j \cdot \alpha_{j-1}} \qquad \beta_j = \frac{\delta_j^n - c_j \beta_{j-1}}{b_j + c_j \cdot \alpha_{j-1}}$$

5) $\alpha_1 - ?$, $\beta_1 - ?$ из ЛГУ

$$\frac{u_2^{n+1} - u_1^{n+1}}{h} = ((n+1)\Delta t)^2$$

$$u_1^{n+1} = u_2^{n+1} - h \cdot \left[((n+1)\Delta t)^2\right]$$

$$u_1^{n+1} = \alpha_1 \cdot u_2^{n+1} + \beta_1 \Rightarrow$$

$$\alpha_1 = 1 \qquad \beta_1 = -h \cdot ((n+1)\Delta t)^2$$

6) $u_{Nx}^{n+1} - ?$

$$u_{Nx}^{n+1} = ((n+1)\Delta t)^2 + 1$$

7) $u_j^{n+1} = \alpha_{j-1} \cdot u_{j+1}^{n+1} + \beta_{j-1}$

Пример 2

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} + 2u + t^2 + x^3 \qquad u(t=0, x) = x^3$$

п. 1-8 примера 1.

$$\begin{cases} \dfrac{\partial u}{\partial x}(t, x=0) = t^2 \\ u(t, x=1) = t^2+1 \end{cases}$$

1) $$\frac{u_j^{n+1}-u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{h^2} + \frac{u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n}}{h} +$$

$$+ u_j^{n+1} + u_j^n + (n\Delta t)^2 + [(j-1)h]^3$$

2) $$u_{j+1}^{n+1}\left[-\frac{\Delta t}{h^2}\right] + u_j^{n+1}\left[1+\frac{2\Delta t}{h^2}-\Delta t\right] + u_{j-1}^{n+1}\cdot\left[-\frac{\Delta t}{h^2}\right] =$$

$$= u_j^n + \Delta t\cdot u_j^n + \Delta t\cdot\left[(n\Delta t)^2 + [(j-1)h]^3\right]$$

3) $$\frac{\Delta t}{h^2} + \frac{\Delta t}{h^2} < 1 + \frac{2\Delta t}{h^2} - \Delta t$$

$$0 < 1 - \Delta t$$

$$\Delta t < 1$$

4) см пример 1.

5) $$u_1^{n+1} = (n+1)\Delta t + 1$$

$$u_1^{n+1} = \alpha_1 \cdot u_2^{n+1} + \beta_1^{n+1}$$

$$u_1^{n+1} = 0 \cdot u_2^{n+1} + (n+1)\Delta t + 1 \Rightarrow$$

$$\alpha_1 = 0 \qquad \beta_1 = (n+1)\Delta t + 1$$

6) $$\frac{u_{N_x}^{n+1} - u_{N_x-1}^{n+1}}{h} = 2{,}7 + (n+1)\Delta t$$

$$u_{N_x-1}^{n+1} = \alpha_{N_x-1} \cdot u_{N_x}^{n+1} + \beta_{N_x-1}$$

$$\frac{u_{N_x}^{n+1} - \alpha_{N_x-1} \cdot u_{N_x}^{n+1} - \beta_{N_x-1}}{h} = 2{,}7 + (n+1)\Delta t$$

$$u_{N_x}^{n+1} = \frac{[2{,}7 + (n+1)\Delta t]h + \beta_{N_x-1}}{1 - \alpha_{N_x-1}}$$

7) $$u_j^{n+1} = \alpha_j \cdot u_{j+1}^{n+1} + \beta_j$$

Д/З: семинар 6

① $$\frac{\partial u}{\partial t} = 10^{-2}\frac{\partial^2 u}{\partial x^2} - 10^{-3}u + tx$$

$$u(t=0, x) = 0$$

$$\begin{cases} \frac{\partial u}{\partial x}(t, x=0) = t \\ \frac{\partial u}{\partial x}(t, x=1) = 2t \end{cases}$$

п. 1-8. примера 1 из семинара 5.

② $$\frac{\partial u}{\partial t} = 10^{-5}\frac{\partial^2 u}{\partial x^2} + 10^{-4}u + t^2$$

$$u(t=0, x) = x$$

$$\begin{cases} u(t, x=0) = t^2 \\ \frac{\partial u}{\partial t}(t, x=1) = t + u(t, x=1) \end{cases}$$

п. 1-8 примера 1.