Двумерные диф.ур-я   Семинар 8
параб. типа
явная схема. Метод расщепления

Условие устойчивости явной схемы:

$\Delta t \leq \frac{h^2}{2\sigma}$

$\frac{\Delta t}{h^2} \leq \frac{1}{2\sigma}$ – для одномерного случая

Для двумерного:

$\frac{\Delta t}{h_x^2} + \frac{\Delta t}{h_y^2} \leq \frac{1}{2\sigma}$ или $\Delta t \leq \frac{1}{\frac{2\sigma}{h_x^2} + \frac{2\sigma}{h_y^2}}$

если $h_x = h_y = h$, то:

$\frac{\Delta t}{h^2} \leq \frac{1}{4\sigma}$

Порядок аппроксимации:

$O(\Delta t, h_x^2, h_y^2)$ для $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$O(\Delta t, h_x^2, h_y)$ для $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Пример 1 явная схема

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 1 + u$$

$u = u(t, x, y)$

$u(t=0, x, y) = xy$

$$\begin{cases} u(t, x=0, y) = y \\ u(t, x=1, y) = y+1 \end{cases}$$

$$\begin{cases} u(t, x, y=0) = x \\ u(t, x, y=1) = x+1 \end{cases}$$

$$\frac{u_{jk}^{n+1} - u_{jk}^{n}}{\Delta t} = \frac{u_{j+1,k}^{n} - 2u_{j,k}^{n} + u_{j-1,k}^{n}}{h_x} + \frac{u_{j,k+1}^{n} - 2u_{j,k}^{n} + u_{j,k-1}^{n}}{h_y} + 1 + u_{jk}^{n}$$

$$u_{jk}^{n+1} = u_{jk}^{n} + \left[\lambda_{xx}^{n} + \lambda_{yy}^{n} + 1 + u_{jk}^{n}\right] \cdot \Delta t$$

н.у.: $u_{jk}^{0} = 0$

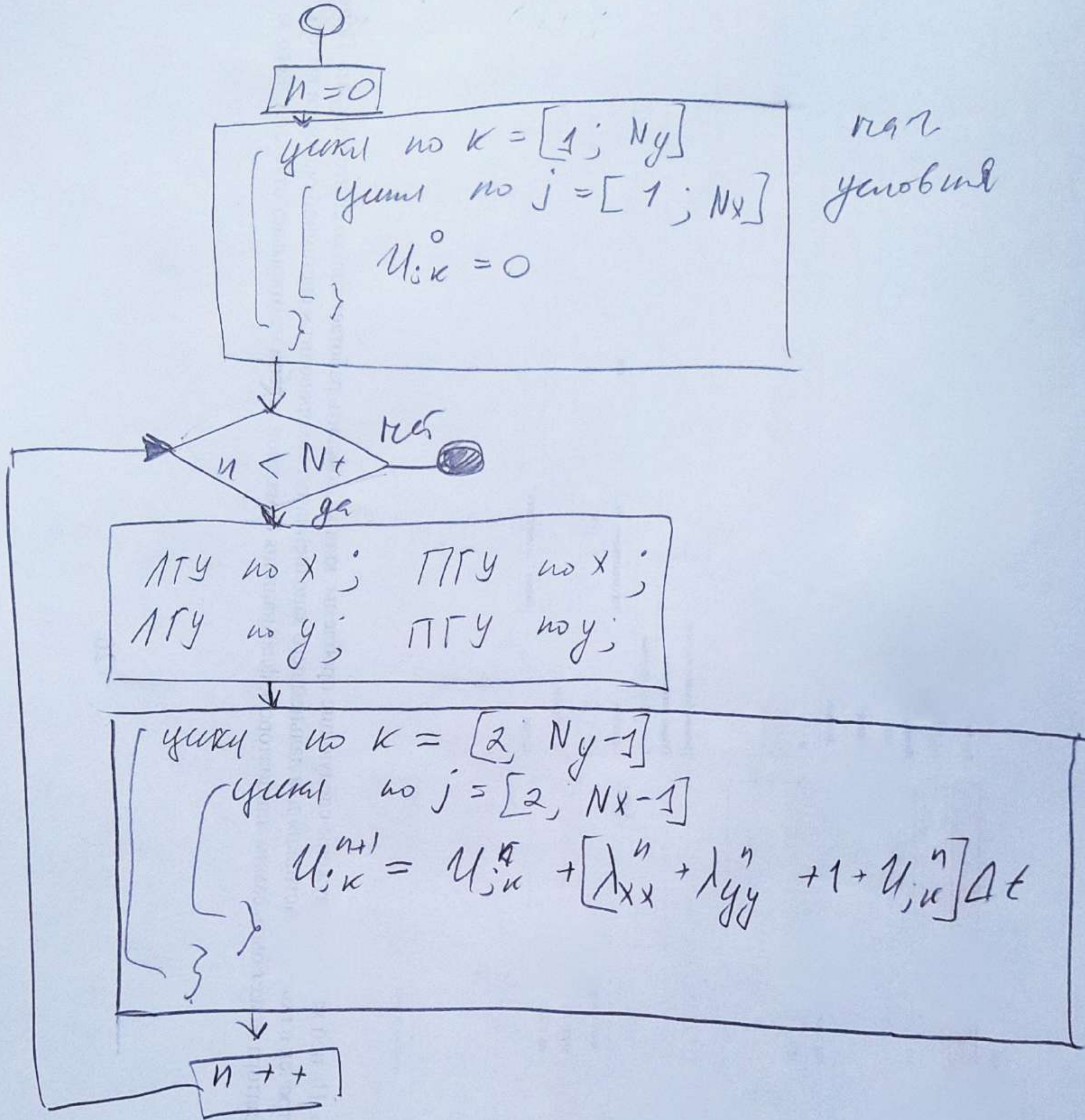
ЛГУ по x: $u_{1,k}^{n+1} = (k-1) h_y$

ПГУ по x: $u_{N_x,k}^{n+1} = (k-1) h_y + 1$

ЛГУ по y: $u_{j,1}^{n+1} = (j-1) h_x$

ПГУ по y: $u_{j,N_y}^{n+1} = (j-1) h_x + 1$

Блок-схема


n = 0
цикл по k = [1; Ny]
цикл по j = [1; Nx]
$U_{jk}^{0} = 0$
нач условия
n < Nt
нет
да
ЛГУ по x; ПГУ по x;
ЛГУ по y; ПГУ по y;
цикл по k = [2, Ny-1]
цикл по j = [2, Nx-1]
$U_{jk}^{n+1} = U_{jk}^{n} + [\lambda_{xx}^{n} + \lambda_{yy}^{n} + 1 + U_{jk}^{n}]\Delta t$
n++

Пример 2 метод расщепления

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0{,}7\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 1$$ (метод дробных шагов)

(1) $$\frac{u_{j,k}^{n+1/2} - u_{j,k}^{n}}{\Delta t} + \frac{u_{j,k}^{n+1/2} - u_{j-1,k}^{n+1/2}}{h_x} = 0{,}7\frac{u_{j+1,k}^{n+1/2} - 2u_{j,k}^{n+1/2} + u_{j-1,k}^{n+1/2}}{h_x^2} + 1$$

(2) $$\frac{u_{j,k}^{n+1} - u_{j,k}^{n+1/2}}{\Delta t} = 0{,}7\frac{u_{j,k+1}^{n+1} - 2u_{j,k}^{n+1} + u_{j,k-1}^{n+1}}{h_y^2}$$

$$O(\Delta t, h_x, h_y^2)$$

(1) Приведем к виду для прогонки:

$$u_{j+1,k}^{n+1/2}\cdot\underbrace{\left[\frac{-0{,}7\Delta t}{h_x^2}\right]}_{a_j} + u_{j,k}^{n+1/2}\cdot\underbrace{\left[1 + \frac{1{,}4\Delta t}{h_x^2} + \frac{1\Delta t}{h_x}\right]}_{b_j} + u_{j-1,k}^{n+1/2}\cdot\underbrace{\left[\frac{-\Delta t}{h_x} - \frac{0{,}7\Delta t}{h_x^2}\right]}_{c_j} = u_{j,k}^{n} + 1$$

$|a_j| + |c_j| < |b_j|$ ✓

$u = u(t, x, y)$

$u(t=0, x, y) = xy$

$$\begin{cases} u(t, x=0, y) = t \\ u(t, x=1, y) = t + y \end{cases}$$

$$\begin{cases} \frac{\partial u}{\partial y}(t, x, y=0) = x \\ \frac{\partial u}{\partial y}(t, x, y=1) = x \end{cases}$$

α_1, β_1 - ? из ЛГУ по x

$u_{1,k}^{n+1/2} = (n+1/2)\Delta t$

$\alpha_1 = 0, \quad \beta_1 = (n+1/2)\Delta t$

$u_{N_x,k}^{n+1/2}$ - ? из ПГУ по x

$u_{N_x,k}^{n+1/2} = (n+1/2)\Delta t + (k-1)h_y$

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② $u_{j,k+1}^{n+1}\cdot\left[\frac{-0{,}7\Delta t}{h_y^2}\right]+u_{j,k}^{n+1}\cdot\left[1+\frac{1{,}4\Delta t}{h_y^2}\right]+u_{j,k-1}^{n+1}\left[-\frac{0{,}7\Delta t}{h_y^2}\right]=u_{j,k}^{n+1/2}$

a_j $\qquad$ b_j $\qquad$ c_j

$|a_j|+|c_j|\leq|b_j|$ ✓

$\tilde{\alpha}_1, \tilde{\beta}_1$ — ? $\qquad$ из ЛГУ по y

$$\frac{u_{j,2}^{n+1}-u_{j,1}^{n+1}}{h_y}=(j-1)h_x$$

$$u_{j,1}^{n+1}=-(j-1)h_x\cdot h_y+u_{j,2}^{n+1}$$

$$\tilde{\alpha}_1=1 \qquad \tilde{\beta}_1=-h_x h_y\cdot(j-1)$$

u_{j,N_y}^{n+1} — ? $\qquad$ из ПГУ по y

$$\frac{u_{j,N_y}^{n+1}-u_{j,N_y-1}^{n+1}}{h_y}=(j-1)h_x$$

$$u_{j,N_y-1}=\tilde{\alpha}_{N_y-1}\cdot u_{j,N_y}+\tilde{\beta}_{N_y-1}$$

$$u_{j,N_y}^{n+1}-\tilde{\alpha}_{N_y-1}\cdot u_{j,N_y}-\tilde{\beta}_{N_y-1}=(j-1)h_x\cdot h_y$$

$$u_{j,N_y}^{n+1}\cdot\left[1-\tilde{\alpha}_{N_y-1}\right]=(j-1)h_x\cdot h_y+\tilde{\beta}_{N_y-1}$$

$$u_{j,N_y}^{n+1}=\frac{\tilde{\beta}_{N_y-1}+(j-1)h_x\cdot h_y}{1-\tilde{\alpha}_{N_y-1}}$$